Let $f(x) = \dfrac{x^2 + x - 6}{9 - x^2}$.

③ EACH UNLESS OTHERWISE NOTED

SCORE: _____ / 55 PTS

[a] Find all intervals on which f is continuous.

f IS RATIONAL, SO f IS CONTINUOUS ON ITS DOMAIN IE. $9 - x^2 \neq 0$
$x \neq \pm 3$

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ ④

[b] Find the limit of f at each discontinuity.
Each limit should be a number, ∞ or $-\infty$. **Write DNE only if the other possibilities do not apply.**

$$\lim_{x \to -3} \frac{x^2 + x - 6}{9 - x^2} = \lim_{x \to -3} \frac{(x+3)(x-2)}{(3+x)(3-x)} = \lim_{x \to -3} \frac{x-2}{3-x} = \frac{-3-2}{3-{-3}} = -\frac{5}{6}$$ ⑥

$$\lim_{x \to 3^+} \frac{x-2}{3-x} = -\infty \quad \text{AND} \quad \lim_{x \to 3^-} \frac{x-2}{3-x} = \infty, \quad \text{SO} \quad \lim_{x \to 3} \frac{x-2}{3-x} \text{ DNE}$$

$\frac{1}{0^-}$ $\frac{1}{0^+}$

[c] State the type of each discontinuity in [b].
Justify your answers by stating which condition of the definition of the discontinuity is satisfied.

$x = -3$ IS A REMOVABLE DISCONTINUITY,
SINCE $\lim_{x \to -3} f(x)$ EXISTS BUT $f(-3)$ DNE ④

$x = 3$ IS AN INFINITE DISCONTINUITY,
SINCE $\lim_{x \to 3^+} f(x) = -\infty$ (OR $\lim_{x \to 3^-} f(x) = \infty$) ④

[d] Find the equations of all horizontal asymptotes of f. ⑤

$$\lim_{x \to \infty} \frac{x^2 + x - 6}{9 - x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{\frac{9}{x^2} - 1} = \frac{1 + 0 - 0}{0 - 1} = -1$$

$$\lim_{x \to -\infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{\frac{9}{x^2} - 1} = \frac{1 + 0 - 0}{0 - 1} = -1$$ ②

$y = -1$ IS THE H.A.

State the Intermediate Value Theorem. **SCORE: _____ / 10 PTS**

IF f IS CONTINUOUS ON $[a,b]$

AND d IS BETWEEN $f(a)$ AND $f(b)$

THEN, FOR SOME $c \in [a,b]$, $f(c)=d$

② EACH

The number of students enrolled at a certain college depends on the cost per unit of classes. **SCORE: _____ / 10 PTS**

Suppose $E = f(c)$, where E is the enrollment at the college, in hundreds of students, and c is the cost per unit, in dollars.

What does $f'(37) = -4$ mean ? Your answer must use all the numbers from that equation, and the correct units for those numbers.

NOTE: Your answer must NOT use "slope", "change" nor "derivative".

IF CLASSES COST \$37 PER UNIT, 2

ENROLLMENT WILL DROP BY 400 STUDENTS, 4

FOR EACH DOLLAR PER UNIT THAT THE TUITION INCREASES 4

Find a function f and a **non-zero** number a such that the derivative of f at a is given by

SCORE: _____ / 10 PTS

$$\lim_{h\to 0}\frac{\sec(h-\pi)+1}{h}$$

Show that your answers are correct using the definition of the derivative at a point.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$a+h=h-\pi \longrightarrow a=-\pi \quad 3\tfrac{1}{2}$$

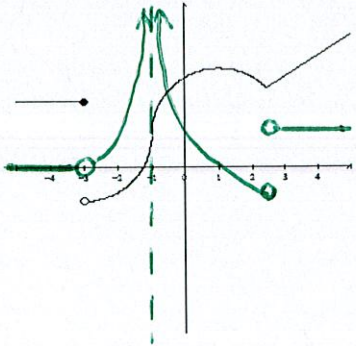
$$f(a+h)=f(-\pi+h)=f(h-\pi)=\sec(h-\pi)\longrightarrow f(x)=\sec x \quad 3\tfrac{1}{2}$$

$$f'(a)=\lim_{h\to 0}\frac{\sec(-\pi+h)-\sec(-\pi)}{h}=\lim_{h\to 0}\frac{\sec(h-\pi)-(-1)}{h}=\lim_{h\to 0}\frac{\sec(h-\pi)+1}{h}$$

3

The graph of $f(x)$ is shown below. Sketch a graph of $f'(x)$ on the same axes.

SCORE: _____ / 15 PTS



(2) $f' = 0$ ON $(-5, -3)$

(1) DNE @ $x = -3$

(2) > 0, INCR ON $(-3, -1)$

(2) $\to \infty$ @ $x = -1$

(2) > 0, DECR ON $(-1, 1)$

(1) $= 0$ @ $x = 1$

(2) < 0, DECR ON $(1, 2.5)$

(1) DNE @ $x = 2.5$

(2) > 0, CONSTANT ON $(2.5, 5)$

If $f(x)=\dfrac{1}{\sqrt{1-x}}$, find $f'(x)$. **SCORE: _____ / 20 PTS**

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\sqrt{1-x-h}}-\frac{1}{\sqrt{1-x}}}{h}\cdot\frac{\sqrt{1-x-h}\,\sqrt{1-x}}{\sqrt{1-x-h}\,\sqrt{1-x}}$$

4

$$= \lim_{h\to 0}\frac{\sqrt{1-x}-\sqrt{1-x-h}}{h\sqrt{1-x-h}\,\sqrt{1-x}}\cdot\frac{\sqrt{1-x}+\sqrt{1-x-h}}{\sqrt{1-x}+\sqrt{1-x-h}}$$

5

$$= \lim_{h\to 0}\frac{(\cancel{1-x})-(\cancel{1-x}-h)}{h\sqrt{1-x-h}\,\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-h}\right)}$$

5

$$= \lim_{h\to 0}\frac{1}{\sqrt{1-x-h}\,\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-h}\right)}$$

$$= \frac{1}{\sqrt{1-x}\,\sqrt{1-x}\left(2\sqrt{1-x}\right)} = \frac{1}{2(1-x)^{3/2}}$$

4 2

At time t minutes, the position of an object moving along a line is $s(t) = \frac{4t}{2+t}$ yards. **SCORE: _____ / 15 PTS**

Find the instantaneous velocity of the object at time $t = 6$. **Give the units of your answer.**

$$\lim_{b \to 6} \frac{s(b) - s(6)}{b - 6}$$

$$= \lim_{b \to 6} \frac{\frac{4b}{2+b} - 3}{b - 6} \cdot \frac{2+b}{2+b}$$

4

$$= \lim_{b \to 6} \frac{4b - (6 + 3b)}{(b-6)(2+b)}$$

$$= \lim_{b \to 6} \frac{\cancel{b-6}}{\cancel{(b-6)}(2+b)}$$

5

$$= \lim_{b \to 6} \frac{1}{2+b} = \frac{1}{8} \text{ yard/minute}$$

4 2

Prove that the equation $x^3 = 4^x - 4$ has a solution in the interval $(-1, 2)$. **SCORE: _____ / 15 PTS**

LET $f(x) = x^3 - 4^x + 4$

f IS CONTINUOUS ON $[-1, 2]$ SINCE f IS A SUM/DIFFERENCE OF POLYNOMIAL + EXPONENTIAL FUNCTIONS WHICH ARE CONTINUOUS EVERYWHERE

$f(-1) = -1 - \frac{1}{4} + 4 > 0$

$f(2) = 8 - 16 + 4 < 0$

SO $f(2) < 0 < f(-1)$ 3

BY IVT, FOR SOME $c \in (-1, 2)$, $f(c) = c^3 - 4^c + 4 = 0$

IE, $c^3 = 4^c - 4$

2 EACH UNLESS OTHERWISE NOTED